Model representation for local energy transfer theory of isotropic turbulence

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Abstract

An almost-Markovian model equation is proposed for Fourier modes of velocity field of isotropic turbulence whose statistical properties are identical to those governed by equations of Local Energy Transfer theory of turbulence [McComb *et. al.*, J. Fluid Mech. **245**, 279 (1992)] compatible with the Kolmogorov spectrum.

1 Introduction

An attempt to solve closure problem of fluid turbulence led Kraichnan to propose Direct Interaction Approximation (DIA) ([Kraichnan(1958)], [Kraichnan(1959)]) as a pioneer renormalized perturbation theory (RPT) followed by other RPTs which have been reviewed from time to time ([Leslie(1973)], [McComb(1990)], [McComb(1995)], [L'vov(1991)], [Lesieur(1997)]). In this paper, our main concern is with Local Energy Transfer (LET) theory of isotropic turbulence which is compatible with Kolmogorov spectrum ([McComb(1978)], [McComb(1990)]). Based on the Edwards's theory ([Edwards(1964)]), the LET was proposed by [McComb(1974)] in an Eulerian framework and, since then, has evolved into a set of equations comprising of

fluctuation-dissipation relation and equations governing the evolution of two-time and single-time velocity correlations of isotropic turbulent flow-field ([McComb(1978)], [McComb *et al.*(1992)]). The LET has been remained under persistent surveillance, especially of McComb and co-workers, for its performance and accomplishments in cases of isotropic turbulence and related passive scalar convection ([McComb *et al.*(1984)], [McComb *et al.*(1989)], [McComb *et al.*(1992)], [Oberlack *et al.*(2001)], [McComb & Quinn(2003)], [Frederiksen *et al.*(1994)], [Frederiksen & Davies(2000)]). The LET's compatibility with Kolmogorov spectrum ([McComb(1990)]) despite its failure to comply with random Galilean invariance ([Kraichnan(1965)]), its encouraging performance and computational simplicity relative to some other RPTs ([McComb(1990)], [Oberlack *et al.*(2001)], [McComb & Quinn(2003)]) are certain niceties of the LET. Further, a long awaited model representation for LET, if exists, would establish the fact that statistical properties predicted by the LET are those of a realizable velocity field.

The model representations are known to exist for a few RPTs formulated in Eulerian framework, such as, DIA ([Kraichnan(1970)]), and Edwards's extended theory ([Kraichnan(1971)]). The DIA equations were associated with a Langevin model equation and an almost-Markovian equation was suggested by [Kraichnan(1971)] and interpreted as a model representation for Edwards's theory ([Edwards(1964)]) when extended to non-steady turbulence cases. Also, model representation exists for Kaneda's theory ([Kaneda(1981)]) in mixed Eulerian-Lagrangian framework put forward by [Kraichnan(1965)]. In the present work, we suggest an existence of an almost-Markovian type model representation for the LET theory.

2 LET theory equations

Consider a homogeneous, isotropic, incompressible fluid turbulence in a reference frame S which is stationary in the laboratory. The Eulerian turbulent velocity field $u_i(\mathbf{x},t)$ in physical space ($\mathbf{x}$) and time (t), with respect to S, can be expressed in terms of Fourier modes $u_i(\mathbf{k},t)$ by

$$u_i(\mathbf{x},t) = \int d^3\mathbf{k} u_i(\mathbf{k},t) \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{1}$$

and which are governed by the Navier-Stokes equation written in wavevector ($\mathbf{k}$) and t domain:

$$(\frac{\partial}{\partial t} + \nu k^2) u_i(\mathbf{k},t) = M_{ijm}(\mathbf{k}) \int d^3\mathbf{p} u_j(\mathbf{p},t) u_m(\mathbf{k}-\mathbf{p},t). \tag{2}$$

Here ν is kinematic viscosity of fluid, inertial transfer operator

$$M_{ijm}(\mathbf{k}) = (2i)^{-1}[k_j P_{im}(\mathbf{k}) + k_m P_{ij}(\mathbf{k})], \tag{3}$$

the projector $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$, $k = |\mathbf{k}|$, and δ_{ij} is the Kronecker delta. The subscripts take the values 1, 2 or 3 alongwith the usual summation convention over repeated subscript. The $u_i(\mathbf{k}, t)$ satisfy continuity condition, i.e. $k_i u_i(\mathbf{k}, t) = 0$ and must satisfy reality requirement $u_i(\mathbf{k}, t) = u_i * (-\mathbf{k}, t)$ where $*$ denotes the complex conjugate. As there is no uniform velocity in each realization of isotropic turbulence, $u_i(\mathbf{0}, t) = 0$. The derivation of statistical properties of the velocity field from equation (2) poses well known turbulence closure problem due to nonlinear term presents on the right hand side. The various RPTs provide ways to tackle the closure problem and discussion here is focused to the solution provided by the LET theory.

The closed set of LET theory equations ([McComb(1978)], [McComb *et al.*(1992)], [McComb & Quinn(2003)]) consists of generalized fluctuation - dissipation relation for the propagator $H_{in}(\mathbf{k}; t, t')$, and equations governing evolution of two-time velocity correlation $Q_{in}(\mathbf{k}, \mathbf{k}'; t, t') = \langle u_i(\mathbf{k}, t) u_n(\mathbf{k}', t') \rangle$ and single-time velocity correlation $Q_{in}(\mathbf{k}, \mathbf{k}'; t, t) = \langle u_i(\mathbf{k}, t) u_n(\mathbf{k}', t) \rangle$ of the velocity field $u_i(\mathbf{k}, t)$ satisfying equation (2). For isotropic turbulence, these statistical properties may be further written as

$$H_{in}(\mathbf{k}; t, t') = P_{in}(\mathbf{k}) H(k; t, t'), \tag{4}$$

$$Q_{in}(\mathbf{k}, \mathbf{k}'; t, t') = P_{in}(\mathbf{k}) Q(k; t, t') \delta(\mathbf{k} + \mathbf{k}'), \tag{5}$$

$$Q_{in}(\mathbf{k}, \mathbf{k}'; t, t) = P_{in}(\mathbf{k}) Q(k; t, t) \delta(\mathbf{k} + \mathbf{k}'), \tag{6}$$

where δ represents Dirac delta function. The LET equations for $H(k; t, t')$, $Q(k; t, t')$ and $Q(k; t, t)$ for isotropic turbulence may be written as

$$Q(k; t, t') = H(k; t, t') Q(k; t', t'), \ \forall t > t' \tag{7}$$

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) Q(k; t, t') = P(k; t, t'), \tag{8}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 \right) Q(k; t, t) = 2P(k; t, t), \tag{9}$$

where the inertial transfer term $P(k; t, t')$ is

$$P(k; t, t') = \int d^3\mathbf{p} L(\mathbf{k}, \mathbf{p}) \Big[\int_0^{t'} ds H(k; t', s) Q(p; t, s) Q(|\mathbf{k} - \mathbf{p}|; t, s) - \int_0^{t} ds H(p; t, s) Q(k; t', s) Q(|\mathbf{k} - \mathbf{p}|; t, s) \Big] \tag{10}$$

and equation (7) represents generalized fluctuation-dissipation relation. Also

$$L(\mathbf{k},\mathbf{p}) = \frac{[\mu(k^2+p^2) - kp(1+2\mu^2)](1-\mu^2)kp}{k^2+p^2-2kp\mu} \tag{11}$$

and μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{p}$. The set of LET equations is compatible with the Kolmogorov spectrum ([McComb(1990)]) and its predictions are encouraging when assessed against the direct numerical simulation results of Navier-Stokes equation (2) ([McComb *et al.*(1984)], [McComb *et al.*(1989)], [McComb *et al.*(1992)], [McComb & Quinn(2003)]). Now we pose a question: Is it possible to obtain a stochastic model equation whose statistical properties are identical to those as predicted by the LET equations (7)-(9)? The answer is affirmative and the model equation is proposed in the section to follow.

3 An almost-Markovian model equation for LET

Consider an almost-Markovian type equation for $u_i(\mathbf{k},t)$, written as

$$\Big(\frac{\partial}{\partial t} + \nu k^2\Big) u_i(\mathbf{k},t) + \alpha(k,t) u_i(\mathbf{k},t) = b_i(\mathbf{k},t) \tag{12}$$

with the statistically sharp damping function $\alpha(k,t)$ and white noise forcing term $b_i(\mathbf{k},t)$ having zero-mean. For a particular choice of $\alpha(k,t)$ and statistical properties of $b_i(\mathbf{k},t)$, this equation (12) can recover LET theory equations. We now obtain that particular choice.

For isotropic turbulence, the evolution of $Q(k;t,t')$ and $Q(k;t,t)$ as predicted by the model equation (12) is governed by the following equations, written as

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) Q(k;t,t') = -\alpha(k,t) Q(k;t,t') \tag{13}$$

and

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) Q(k;t,t) = -2\alpha(k,t) Q(k;t,t) + B(k,t). \tag{14}$$

Here $B(k,t)$ is statistical property of the white noise forcing term $b_i(\mathbf{k},t)$ and satisfies

$$\langle b_i(\mathbf{k},t) b_n(\mathbf{k}',t')\rangle = P_{in}(\mathbf{k}) B(k,t) \delta(\mathbf{k}+\mathbf{k}') \delta(t-t'). \tag{15}$$

Comparing equations (13) and (14) to the equations (8) and (9), respectively, suggests

$$\alpha(k,t) = -\frac{P(k;t,t')}{Q(k;t,t')} \tag{16}$$

and

$$B(k,t) = P(k;t,t) - P(k;t,t')\frac{Q(k;t,t)}{Q(k;t,t')}. \tag{17}$$

It should be noted that the model equation's response function $H(k;t,t')$ is governed by

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) H(k;t,t') = -\alpha(k,t)H(k;t,t') \; \forall \, t > t' \tag{18}$$

and together with equation (13) suggests $Q(k;t,t') = H(k;t,t')Q(k;t',t')$ identical to generalized fluctuation-dissipation relation (7) used in LET. Thus the equation (12), along with the expression for α given by (16) and statistical properties of white noise forcing term satisfying (15) and (17), is almost-Markovian model representation which can be associated with LET theory.

4 Concluding remarks

The purpose of this paper has been to propose an almost-Markovian model representation for local energy transfer (LET) theory. The proposed Markovian model equation can be used to generate velocity field whose statistical properties are identical to those of LET theory. And in that respect, prediction of LET theory are for the velocity field that is realizable and governed by the Markovian model equation. It should be worth mentioning here that an another choice for $\alpha(k,t)$ and $b_i(\mathbf{k},t)$ resulted in an almost-Markovian model representation for extended Edwards's theory ([Kraichnan(1971)]). In fact, the self-consistent Edwards's theory provided the foundation for LET theory ([McComb & Quinn(2003)]). And not to our surprise, both LET and extended Edwards's theories can be associated with the Markovian type model equation where generalized fluctuation-dissipation relation is central.

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References

[Edwards(1964)] Edwards, S. F. 1964 The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239–273.

[Frederiksen & Davies(2000)] Frederiksen, J. S. & Davies, A. G. 2000 Dynamics and spectra of cumulant update closure for two-dimensional turbulence. *Geophys. Astrophys. Fluid Dyn.* **92**, 197–231.

[Frederiksen *et al.*(1994)] Frederiksen, J. S., Davies, A. G. & Bell, R. C. 1994 Closure theories with non-Gaussian restarts for truncated two-dimensional turbulence. *Phys. Fluids* **6**, 3153–3163.

[Kaneda(1981)] Kaneda, Y. 1981 Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function. *J. Fluid Mech.* **107**, 131–145.

[Kraichnan(1958)] Kraichnan, R. H. 1958 Irreversible statistical mechanics of incompressible hydromagnetic turbulence. *Phys. Rev.* **109**, 1407–1422.

[Kraichnan(1959)] Kraichnan, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–543.

[Kraichnan(1965)] Kraichnan, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575–598.

[Kraichnan(1970)] Kraichnan, R. H. 1970 Convergents to turbulence functions. *J. Fluid Mech.* **41**, 189–217.

[Kraichnan(1971)] Kraichnan, R. H. 1971 An almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513–524.

[Lesieur(1997)] Lesieur, M. 1997 *Turbulence in Fluids*, 3rd edn. Dordrecht: Kluwer.

[Leslie(1973)] Leslie, D. C. 1973 *Developments in the Theory of Turbulence*. Oxford: Clarendon Press.

[L'vov(1991)] L'vov, V. S. 1991 Scale invariant-theory of fully-Developed hydrodynamic turbulence - Hamiltonian approach. *Phys. Rep.* **207**, 1–47.

[McComb(1974)] McComb, W. D. 1974 A local energy-transfer theory of isotropic turbulence. *J. Phys. A* **7**, 632–649.

[McComb(1978)] McComb, W. D. 1978 A theory of time-dependent isotropic turbulence. *J. Phys. A* **11**, 613–632.

[McComb(1990)] McComb, W. D. 1990 *The Physics of Fluid Turbulence*. New York, NY: Oxford University Press.

[McComb(1995)] McComb, W. D. 1995 Theory of turbulence. *Rep. Prog. Phys.* **58**, 1117–1206.

[McComb *et al.*(1992)] McComb, W. D., Filipiak, M. J. & Shanmugasundaram, V. 1992 Rederivation and further assesment of the LET theory of isotropic turbulence, as applied to passive scalar convection. *J. Fluid Mech.* **245**, 279–300.

[McComb & Quinn(2003)] McComb, W. D. & Quinn, A. P. 2003 Two-point, two-time closures applied to forced isotropic turbulence. *Physica A* **317**, 487–508.

[McComb *et al.*(1984)] McComb, W. D. & Shanmugasundaram, V. 1984 Numerical calculations of decaying isotropic turbulence using the LET theory. *J. Fluid Mech.* **143**, 95–123.

[McComb *et al.*(1989)] McComb, W. D., Shanmugasundaram, V. & Hutchinson, P. 1989 Velocity derivative skewness and two-time velocity correlations of isotropic turbulence as predicted by the LET theory. *J. Fluid Mech.* **208**, 91–114.

[Oberlack *et al.*(2001)] Oberlack, M., McComb, W. D. & Quinn, A. P. 2001 Solution of functional equations and reduction of dimension in the local energy transfer theory of incompressible, three-dimensional turbulence. *Physical Review E* **63**, 026308–1–026308–5.